ON THE PROBLEM OF GAMES ENCOUNTER OF MOTIONS

PMM Vol. 32, №5, 1968, pp. 793-803
N. N. KRASOVSKII
(Sverdlovsk)
(Received April 18, 1968)

A method of formal justification of an extremal aiming scheme [4] for the problem of games encounter of motions [1 – 4] is described.

1. Let us consider the problem of encounter of pursuing ($y[t]$) and pursued ($z[t]$) motions described by the differential equations

$$dy/dt = f^{(1)}[y, u] \tag{1.1}$$

$$dz/dt = f^{(2)}[z, v] \tag{1.2}$$

Here y, z are the $n^{(1)}$- and $n^{(2)}$-dimensional vectors of the objects, respectively; u, v are the $r^{(1)}$- and $r^{(2)}$-dimensional vectors of the controlling forces; $f^{(i)}$ are continuous and differentiable vector functions. The realizations $u[t]$ and $v[t]$ of the permissible controls are restricted by the conditions

$$u[t] \in U, \quad v[t] \in V \tag{1.3}$$

where U, V are closed bounded sets in $r^{(1)}$- and $r^{(2)}$-dimensional spaces, respectively. Encounter of the motions $y[t]$ and $z[t]$ at the instant $t = t_*$ is defined by the equality

$$\{y[t_*]\}_m = \{z[t_*]\}_m \tag{1.4}$$

where $\{y\}_m$ and $\{z\}_m$ are vectors consisting of the first m ($m \leqslant n^{(i)}$) coordinates of the vectors y and z. (The vectors considered below are assumed to be vector columns unless otherwise stipulated (e.g. unless transposition is mentioned).

The problem consists in choosing a permissible control u, which ensures encounter of the motions $y[t]$ and $z[t]$ whatever the piecewise-continuous realization $v[t]$ satisfying condition (1.3) (we are referring to the chosen domain of possible initial conditions $y[t_0]$, $z[t_0]$). In an earlier paper [4] we suggested that this problem be solved by a rule for constructing u, which we shall call "extremal aiming". For linear systems this rule is discussed in [5]. It has been noted that its use and justification involves difficulties. One of these has to do with the fact that extremal aiming generally defines the control $u^\circ[t] = u^\circ(y[t], z[t])$ as a non-singlevalued and discontinuous function $u(y, z)$ (e.g. see [6]).

The first of these two difficulties can be circumvented by limiting attention to those cases where the target point $q^\circ[t]$ which determines the optimal control is unique. This leaves us with a fairly rich class of problems to be investigated. However, if we then exclude problems in which discontinuous functions u° can occur, the remaining class of problems is probably unjustifiably meager. It is therefore advisable to investigate extremal control under the condition of a unique target point $q^\circ(t)$ without any further narrowing of the class of problems. However, in this case, it is necessary to work on Eq. (1.1) which contains the discontinuous function $u = u^\circ(y, z)$. This approach obliges us to use generalized solutions [7] of such equations.

The generalized solutions $y[t]$, $z[t]$ enable us to overcome the difficulties involved in the problem of existence of an extremal motion $y[t]$. A simple example shows, however, that in a sufficiently natural class of generalized solutions $y(t)$ the extremal control $u^\circ(y[t], z[t])$ does not ensure encounter of the motions $y[t]$ and $z[t]$ in every case.

In fact, let us consider system (7) of [6]. We regard as a solution $y[t]$ of the corresponding equation $\frac{dy_1}{dt} = y_2, \quad \frac{dy_2}{dt} = -y_1 + u^\circ(y, z[t])$

and absolutely continuous vector function $y[t]$ which satisfies conditions

$$\frac{dy_1[t]}{dt} = y_2[t], \qquad \frac{dy_2[t]}{dt} = -y_1[t] + u_0$$

for almost all values $t \geqslant t_0$. Here $u_0 = u^\circ(y[t], z[t])$ at the points of continuity of the function $u^\circ(y, z)$, and u_0 is any number from the interval $-\mu \leqslant u_0 \leqslant \mu$ at the points $y = y[t]$, $z = z[t]$, where the function $u^\circ(y, z)$ is discontinuous. In the case $\mu - \nu = 1$, $\nu > 2$, system (7) of [6] interpreted in this way for $v[t] = 2$ has the generalized solution $y_1[t] - z_1[t] = -2$, $y_2[t] - z_2[t] = 0$ $(u_0 = 0)$ which defines motions $y[t]$ and $z[t]$ which do not converge with increasing time t.

Thus, the extremal aiming rule in the class of generalized solutions $y[t]$ and $z[t]$ requires improvement. This improvement, which includes a braking constraint on the value of the instant of absorption [4 and 5] $t^\circ = t + \vartheta^\circ[t]$ and is based on a time-discrete computational scheme, is described in [6] for the case of linear monotype objects; the same improvement is justified in [8] for the general case of linear systems. The purpose of the present paper is to present an analogous improvement for the extremal aiming rule, which also includes a new constraint for the quantity $\vartheta^\circ[t]$, but which will be treated within the framework of generalized solutions of the above differential equations with a discontinuous right side. We emphasize that this modification of the problem is formal in character. With computer realization its expansion can take the form of the approximating scheme described in [8].

2. Let us define the improved extremal control u°. Its construction rests on an ancillary construction which is compared with the realized states $y[t]$ and $z[t]$ at each instant t.

We begin with some preliminary remarks. Let t be some temporarily fixed instant. As usual, we apply the term "attainability domain" $H^{(1)}[y, \vartheta]$ (for the motion $y(\tau)$ from the state $y[t] = y$ by the instant $\tau = t + \vartheta$) the set of those and only those points $\{y\}_m$ in m-dimensional space to which the system $dy/d\tau = f^{(1)}[y, u]$ can be brought in the time $t \leqslant \tau \leqslant t + \vartheta$ from the given state $y[t] = y$ through the choice of the program control $u(\tau)$ $(t \leqslant \tau < t + \vartheta)$ restricted by the condition $u(\tau) \in U$. The attainability domain $H^{(2)}[z, \vartheta]$ is similarly defined.

If system (1.1) is defined by the linear equation

$$dy / dt = A^{(1)}y + B^{(1)}u \tag{2.1}$$

where restriction (1.3) is of the form

$$\| u[t] \| \leqslant \mu \qquad (\mu > 0 - \text{const}) \tag{2.2}$$

where the symbol $\| u \|$ represents the Euclidean norm of the vector u, then we know (e.g. see [5]) that the domain $H^{(1)}[y, \vartheta]$ consists of those and only those points $q = \{y\}_m$, which satisfy the inequality

$$\mu \int_0^\vartheta \| \{Y[\vartheta, \tau] B^{(1)}\}_m' l \| d\tau + \{Y[\vartheta, 0] y\}_m' l - l'q \geqslant 0 \tag{2.3}$$

for any choice of the m-dimensional vector l. Here the symbol $Y[\tau, \tau_0]$ denotes the fundamental matrix of solutions for the equation $dy / d\tau = A^{(1)}y$; the prime signifies transposition; the symbol $\{Q\}_m$ stands for a matrix consisting of the first m rows of the matrix Q. Because of the inhomogeneity of the left side of (2.3) in l, it is sufficient (and necessary) that condition (2.3) be fulfilled only for some suitable subset L of vectors l, e.g. for all l with the norm $\| l \| \leqslant 1$ or $- \| l \| = 1$, etc. Similarly, in the case of the linear equation

$$dz / dt = A^{(2)}z + B^{(2)} v \tag{2.4}$$

under the restriction

$$\| v [t] \| \leqslant \nu \qquad (\nu > 0 - \text{const}) \tag{2.5}$$

the domain $H^{(2)}[z, \vartheta]$, like (2.3), is described by the inequality

$$\nu \int_0^{\vartheta} \| \{Z[\vartheta, \tau] B^{(2)}\}_m' l \| d\tau + \{Z[\vartheta, 0] z\}_m' l - l'p \geqslant 0 \tag{2.6}$$

which must be fulfilled for every point p from $H^{(2)}[z, \vartheta]$ for all l (for all l from L).

We assume that the domains $H^{(1)}[y, \vartheta]$ are convex closed sets. Along with the attainability domains $H^{(2)}[z, \vartheta]$ we shall consider certain associated bounded convex closed sets $G^{(2)}[z, \vartheta]$ containing them, so that for all of the values of z and ϑ under consideration we have the inclusions

$$H^{(2)}[z, \vartheta] \subset G^{(2)}[z, \vartheta] \tag{2.7}$$

Here we assume that the sets $G^{(2)}[z, \vartheta]$ also satisfy the condition

$$G^{(2)}[z[t^*], \vartheta - t^*] \subset G^{(2)}[z[t_*], \vartheta - t_*] \quad \text{при} \quad t^* > t_* \tag{2.8}$$

whatever the motion $z[t]$ of system (1.2) generated by the permissible control $v[t]$ $(t_* \leqslant t < t^*)$. This condition is fulfilled automatically if $H^{(2)}[z, \vartheta] = G^{(2)}[z, \vartheta]$.

The sets $G^{(1)} = H^{(1)}$ and $G^{(2)}$ are the intersections of their basis half-spaces [9]. We shall therefore describe them by means of inequalities similar to conditions (2.3) and (2.6). Hence, let the domains $G^{(1)}[y, \vartheta]$ and $G^{(2)}[z, \vartheta]$ be described by the inequalities (Fig. 1)

$$\rho^{(1)}[y, \vartheta, l] - l'q \geqslant 0, \quad \rho^{(2)}[z, \vartheta, l] - l'p \geqslant 0$$

respectively, which must be fulfilled for $q \in G^{(1)}$ and $p \in G^{(2)}$ for all l ($\| l \| = 1$). We shall assume that functions $\rho^{(i)}$ which are convex and homogeneous in l are continuous and that they satisfy the Lipschits conditions.

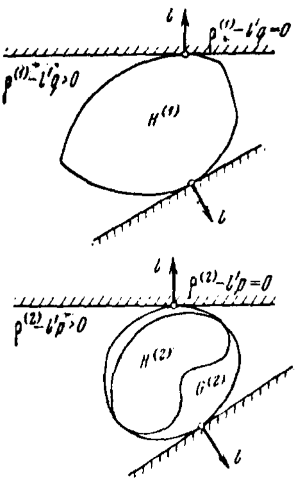

Fig. 1

The instant $t^\circ = t + \vartheta^\circ(y[t], z[t]) \geqslant t$, when the condition

$$G^{(2)}[z[t], \vartheta^\circ] \subset G^{(1)}[y[t], \vartheta^\circ]$$

is fulfilled for the first time will be called the "instant of absorption" of process (1.2) by process (1.1) (for a given initial state $y[t]$, $z[t]$). In order to construct the extremal control u° we also need to consider the closed ε-neighborhoods of the domains $G^{(1)}[y, \vartheta]$. We denote these ε-neighborhoods by the symbol $H^{(1)}[y, \vartheta; \varepsilon]$.

In the case of a linear system (2.4) under convex restriction (2.5) we can set $G^{(2)} = H^{(2)}$. Further, in case of a linear system (2.1) under restriction (2.2) the domain $G^{(1)}[y, \vartheta; \varepsilon]$ consists by definition of those and only those points p for each of which there exists a point q satisfying conditions (2.3) and the inequality $\| q - p \| \leqslant \varepsilon$. This means that the domain $G^{(1)}[y, \vartheta; \varepsilon]$ contains those and only those points p, for which the condition

$$\max_{s} \min_{l} \left(\mu \int_0^{\vartheta} \| \{Y[\vartheta, \tau] B^{(1)}\}_m' l \| d\tau + \{Y[\vartheta, 0] y\}_m' l - l'(p + s)\right) \geqslant 0$$

$$\text{for } \|s\| \leqslant \varepsilon, \ \|l\| \leqslant 1 \qquad (2.9)$$

is fulfilled ($s = q - p$).

The operations max and min can be interchanged [?]. But then (2.9) immediately implies that the domain $G^{(1)}[y, \vartheta; \varepsilon]$ is described by the inequality

$$\varepsilon + \mu \int_0^{\vartheta} \| \{Y[\vartheta, \tau] B^{(1)}\}_m' l \| d\tau + \{Y[\vartheta, 0] y\}_m' l - l' p \geqslant 0 \qquad (2.10)$$

which must be fulfilled for every point $p \in G^{(1)}$ for all l with the norm $\| l \| = 1$. Finally, the condition of absorption of the domain $G^{(2)}[z, \vartheta]$ by the domain $G^{(1)}[y, \vartheta; \varepsilon]$ becomes

$$\varepsilon + \mu \int_0^{\vartheta} \| \{Y[\vartheta, \tau] B^{(1)}\}_m' l \| d\tau - \nu \int_0^{\vartheta} \| \{Z[\vartheta, \tau] B^{(2)}\}_m' l \| d\tau + \{Y[\vartheta, 0] y -$$

$$- Z[\vartheta, 0] z\}_m' l \geqslant 0 \qquad (\|l\| = 1) \qquad (2.11)$$

since here condition (2.10) must follow from condition (2.6) (for the same value $p \in G^{(2)}[z, \vartheta]$).

In the general case the domain $G^{(1)}[y, \vartheta; \varepsilon]$ is described by the inequality

$$\varepsilon + \rho^{(1)}[y, \vartheta, l] - l' p \geqslant 0 \qquad (\|l\| = 1) \qquad (2.12)$$

while the condition of absorption of the domain $G^{(2)}[z, \vartheta]$ by the domain $G^{(1)}[y, \vartheta; \varepsilon]$ becomes

$$\varepsilon + \rho^{(1)}[y, \vartheta, l] - \rho^{(2)}[z, \vartheta, l] \geqslant 0 \quad (\|l\| = 1) \qquad (2.13)$$

Now let us turn from these ancillary remarks to the actual construction of the extremal control u°. To do this we consider the $(n^{(1)} + n^{(2)} + 1)$-dimensional phase space W whose elements are the triplets $\{y, z, \vartheta\}$, where ϑ is a scalar variable, $\vartheta > 0$. We break down the whole space W into two parts, W_θ and W^ε.

The set W_θ consists of those and only those triplets $\{y, z, \vartheta\}$ for which $\vartheta \geqslant \vartheta^\circ(y, z)$; the set W^ε on the other hand, consists of all those triplets $\{y, z, \vartheta\}$ for which $\vartheta < \vartheta^\circ(y, z)$.

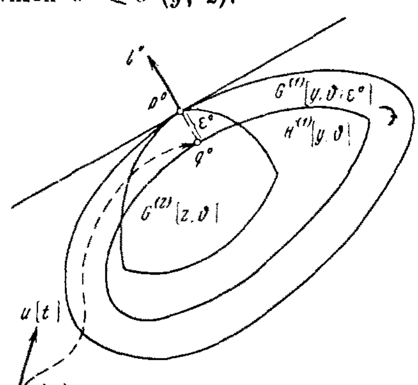

Fig. 2

We shall construct the control u° as a function of the quantities y, z, ϑ, so that the realized value of the control $u^\circ[t]$ is given by Eq. $u^\circ[t] = u^\circ(y[t], z[t], \vartheta[t])$. Here the variation of the vector functions $y[t]$ and $z[t]$ is defined by Eqs. (1.1) and (1.2); the law of variation of the additional variable $\vartheta[t]$ will be given below. In the domain W_0 the control u° is a non-singlevalued function which can assume any values satisfying the condition

$$u^\circ \in U \qquad (2.14)$$

The control u° in the domain W^ε is constructed as follows. Let $\varepsilon^\circ(y, z, \vartheta)$ be the smallest value of $\varepsilon > 0$ for which the domain $G^{(1)}[y, \vartheta; \varepsilon]$ contains the domain $G^{(2)}[y, z]$. (By the definition of the set W^ε we have $\varepsilon^\circ(y, z, \vartheta) > 0$ if $\{y, z, \vartheta\} \in W^\varepsilon$.)

We say that the domain $G^{(1)}$ absorbs the domain $G^{(2)}$ regularly if the boundaries of these domains have a single common point p°. We assume that this basic condition is fulfilled in all cases below. Such a situation will be called the "regular case".

Let q° be the point of the domain $G^{(1)}[y, \vartheta]$ which is closest to the point p°. We denote by the symbol $u_*(y[t], z[t], \vartheta[t])$ the control $u_*[t]$ which at the instant t aims the motion y at the point

$$\{y(t+\vartheta)\}_m = q^\circ$$

In other words $u_*[t] = u_*(t)$, where $u_*(\tau)$ $(t \leqslant \tau < t+\vartheta)$ is the progam control which brings the system $dy/d\tau = f^{(1)}[y, u]$ from the state $y[t]$ to the state $\{y(t+\vartheta)\}_m = q^\circ$ (Fig. 2).

The function $u_*(t)$ satisfies the conditions of the maximum principle [10]. We assume that the function $u^\circ(y, z, \vartheta) = u_*[t]$ is generally non-singlevalued at the point y, z, ϑ, where it can assume any values which satisfy the conditions of the maximum principle.

Thus, the control $u^\circ(y, z, \vartheta)$ has been defined for all values $\{y, z, \vartheta\}$ from W. We must now complement system (1.1), (1.2) with relations defining the variation of the quantity $\vartheta[t]$. We assume that the function $\vartheta[t]$ (which can generally be discontinuous) is continuous in W^ε and that it satisfies Eq.

$$d\vartheta/dt = -1, \{y, z, \vartheta\} \in W^\varepsilon \tag{2.15}$$

In the domain W_0 this function satisfies the inequality

$$(d\vartheta/dt)^{(b)} \leqslant -1, \ \{y, z, \vartheta\} \in W_0 \tag{2.16}$$

Here the symbol $(d\vartheta/dt)^{(b)}$ denoted the upper derivative.

The term "generalized solution" $y[t], z[t], \vartheta[t]$ $(t_0 \leqslant t < T)$ of system (1.1), (1.2), (2.15), (2.16) for $u = u^\circ(y, z, \vartheta)$ will be applied to any vector function $\{y[t], z[t], \vartheta[t]\}$ $(t_0 \leqslant t < T)$ which satisfies the following conditions:

1) The vector function $z[t]$ is continuous for all $t \in [t_0, T)$ satisfies ordinary differential equation (1.2), where $v = v[t]$. Here the symbol dz/dt in (1.2) represents the right-hand derivative; any realization of $v[t]$ which is continuous on the right and is restricted by condition (1.3) is acceptable.

2) The vector function $y[t]$ is absolutely continuous and for almost all $t \in [t_0, T)$ satisfies Eq. (1.1), where $u = u^\circ(y, z, \vartheta)$ and $z = z[t]$, $\vartheta = \vartheta[t]$.

3) The function $\vartheta[t]$ for all $t \in [t_0, T)$ is continuous on the right and satisfies conditions (2.15), (2.16). Here the condition $\varepsilon^\circ(y[t], z[t], \vartheta[t]) = 0$ must be fulfilled in the domain W_0.

The vector functions $\{y[t], z[t], \vartheta[t]\}$ which are generalized solutions of system (1.1), (1.2), (2.15), (2.16) will also be called the "motions" of this system generated by the controls $u^\circ(y, z, \vartheta)$ and $v[t]$.

3. Let us consider the properties of the extremal control $u^\circ(y, z, \vartheta)$ defined in the preceding section. First, we note that the domain W^ε is an open set.

In fact, let the point $\{y_*, z_*, \vartheta_*\} \in W^\varepsilon$. Then $\varepsilon^\circ(y_*, z_*, \vartheta) > 0$ for all $0 \leqslant \vartheta \leqslant \vartheta_*$, and in accordance with (2.13) we have

$$\min_{\|l\|=1} (\varepsilon^\circ(y_*, z_*, \vartheta) + \rho^{(1)}[y_*, \vartheta, l] - \rho^{(2)}[z_*, \vartheta, l]) = 0 \quad (0 \leqslant \vartheta \leqslant \vartheta_*) \tag{3.1}$$

The functions $\rho^{(i)}$ are continuous. This implies that for a sufficiently small $\delta > 0$ we have the inequality

$$\rho^{(1)}[y, \vartheta, l] - \rho^{(2)}[z, \vartheta, l] \leqslant -\alpha \qquad (\alpha > 0 - \text{const}) \tag{3.2}$$

for all $0 \leqslant \vartheta \leqslant \vartheta_* + \delta$ and $\|l\| = 1$, provided that $\|y - y_*\| \leqslant \delta$, $\|z - z_*\| \leqslant \delta$. But according to (2.13) this means that the inequality $\varepsilon^\circ(y, z, \vartheta) \geqslant \alpha$, i.e. that $\vartheta^\circ(y, z) > \vartheta_*$ is valid for such y, z and ϑ which means that some neighborhood of the point $\{y_*, z_*, \vartheta_*\}$ does, in fact, lie in the domain W^ε.

Let us consider some point $\{y_*, z_*, \vartheta_*\} \in W^\varepsilon$. Let $u_*(\tau)$ $(t = 0 \leqslant \tau < \vartheta_*)$ be the program control which determines the function $u_*(y_*, z_*, \vartheta_*)$ which generates the extremal control $u^\circ(y_*, z_*, \vartheta_*)$ (see the beginning of Section 2). (The initial instant from which time is measured does not matter in the steadystate systems under consideration, so that we can set $t = 0$.) The control $u_*(\tau)$ brings the system

$$dy/d\tau = f^{(1)}[y, u] \tag{3.3}$$

to the point $\{y(\vartheta_*)\} = q^\circ$ lying on the boundary of the attainability domain $G^{(1)}[y_*, \vartheta_*]$. The function $u_*(\tau)$ therefore satisfies the conditions of the maximum principle [10],

$$H(\psi(\tau), y(\tau), u_*(\tau)) = \max_{u \in U} H(\psi(\tau), y(\tau), u) \qquad (0 \leqslant \tau \leqslant \vartheta_*) \tag{3.4}$$

Moreover, the vector function $\psi(\tau)$ satisfies the boundary condition

$$\psi_j(\vartheta_*) = l_j^\circ, \qquad \psi_i(\vartheta_*) = 0 \qquad (j = 1, \ldots, m;\ i = m+1, \ldots, n^{(1)}) \tag{3.5}$$

where l° is precisely that m-vector l, for which, in accordance with (2.13) and (3.1), we have

$$\min_{\|l\|=1} [\varepsilon^\circ(y_*, z_*, \vartheta_*) + \rho^{(1)}[y_*, \vartheta_*, l] - \rho^{(2)}[z_*, \vartheta_*, l]] =$$
$$= \varepsilon^\circ(y_*, z_*, \vartheta_*) + \rho^{(1)}[y_*, \vartheta_*, l^\circ] - \rho^{(2)}[z_*, \vartheta_*, l^\circ] = 0 \tag{3.6}$$

It is important to note that in the regular case (which we are considering here) the vector l° also varies continuously with changes in y_*, z_* and ϑ_* This is a consequence of the continuity of the functions $\rho^{(i)}$ and of the uniqueness of this vector (for given y_*, z_*, ϑ_*), which in turn follows from the uniqueness of the point p°. Under sufficiently general assumptions (which always hold for linear system (2.1) the continuous variation of ϑ_* and of the vector $\psi(\vartheta_*)$ as well as of the target point q_*) results in continuous variation of the vector $\psi(\tau)$ $(0 \leqslant \tau \leqslant \vartheta_*)$. We assume that this condition is fulfilled. Hence, in the regular case the vector $\psi(0)$ in maximum condition (3.4) varies continuously (in the domain W^ε) with continuous variation of y_*, z_*, ϑ_*. Thus, the control $u^\circ(y_*, z_*, \vartheta_*)$ is determined by the maximum condition

$$\psi'(0) f^{(1)}[y_*, u^\circ] = \max_{u \in U} (\psi'(0) f^{(1)}[y_*, u]) \tag{3.7}$$

The closed sets $U(y_*, \psi(0))$ which determine the values of the function $u^\circ(y_*, z_*, \vartheta_*)$ are, according to condition (3.7), semicontinuous above (in y_* and $\psi(0)$) relative to inclusion.

We shall assume that the sets R through which the vector $f^{(1)}$ as u runs through U are convex [11].

This requirement is again necessarily fulfilled in the case of linear system (2.1) under restriction (2.2).

In fact, maximum condition (3.7) in this case can be written as

$$l^{\circ\prime}\{Y[\vartheta_*, 0] B^{(1)}\}_m u_* = \max_{\|u\| \leqslant \mu} l^{\circ\prime}\{Y[\vartheta_*, 0] B^{(1)}\}_m u \tag{3.8}$$

It does not degenerate if and only if

$$\|Y[\vartheta_*, 0] B^{(1)}\}_m' l^\circ\| > 0 \tag{3.9}$$

whereupon

$$u_* = \mu \frac{\{Y[\vartheta_*, 0]B^{(1)}\}_m' l^\circ}{\|\{Y[\vartheta_*, 0]B^{(1)}\}_m' l^\circ\|} \tag{3.10}$$

We must now discuss the existence of a generalized solution $y[t]$, $z[t]$, $\vartheta[t]$ if system (1.1),(1.2),(2.15),(2.16). Here again the existence of functions $y[t]$,$z[t]$, $\vartheta[t]$ satisfying conditions (1) – (3) (until the instant of encounter or until $T = \infty$ if there is no encounter) under sufficiently general conditions (see [11]) (which are in any case fulfilled for linear system (2.1) under restrictions (2.2)) can be verified, for example, by taking the limit of an approximating discrete scheme similar to that described in [8].

Without discussing the situation in the general case, and assuming that the conditions of existence of generalized solutions (1) – (3) are fulfilled, let us consider briefly the proof of the existence of the required functions $y[t]$, $z[t]$, $\vartheta[t]$ only for the case of linear system (2.1) under convex restriction (2.2). Here we construct a sequence ($s = 1, 2, \ldots$) of descrete schemes described in [8]; the interval $\delta_s = \tau_{k+1} - \tau_k$ of these schemes tends to zero as $s \to \infty$. We then verify that the sequence $\{y^{(s)}[t], z[t], \vartheta^{(s)}[t]\}$ of solutions generated by the corresponding descrete schemes contains a subsequence $\{y^{(i)}[t], z[t], \vartheta^{(i)}[t]\}$ $(i = s_j, j = 1,2,\ldots)$ which converges in the appropriate fashion to the limiting element $\{y[t], z[t], \vartheta[t]\}$ $(t_0 \leqslant t < T)$ which constitutes the required solution of system (2.1),(2.2),(2.15),(2.16). The possibility of constructing the subsequence $\{y^{(i)}[t], z[t], \vartheta^{(i)}[t]\}$ is determined by the following facts: for $t_0 \leqslant t < T$ the set of functions $u[t]$ restricted essentially by condition (2.2) which contains the functions $u^{(s)}[t]$ is weakly compact; the functions $y^{(s)}[t]$ are uniformly bounded and satisfy the Lipschitz conditions in equal degree. This means that we can choose from among them a subsequence which converges uniformly to the required function $y[t]$; the functions $\vartheta^{(s)}[t] + t$ are monotonically nonincreasing, so that from among them we can isolate subsequences which converge essentially to the appropriate function $t + \vartheta[t]$.

4. We must now verify our hypothesis whereby the extremal control $u^\circ(y, z, \vartheta)$ ensures encounter of the motions $y[t]$ and $z[t]$, provided that the state $y[t_0]$, $z[t_0]$ at the initial instant $t = t_0$ is such that an instant of absorption $t^\circ = t_0 + \vartheta^\circ(y[t_0], z[t_0])$ exists.

To prove the validity of this hypothesis concerning encounter of the motions $z[t]$ and $y[t]$ we need merely show that for any motion $\{y[t], z[t], \vartheta[t]\}$ satisfying the initial condition

$$\{y[t_0],\ z[t_0],\ \vartheta[t_0]\} \in W_0 \tag{4.1}$$

we have the inclusion

$$\{y[t],\ z[t],\ \vartheta[t]\} \in W_0 \tag{4.2}$$

for all $t \geqslant t_0$ (until encounter).

In fact, let the initial state of system (1.1),(1.2) at the instant $t = t_0$ be defined by the phase vectors $y = y[t_0]$, $z = z[t_0]$. We set the initial value $\vartheta[t_0]$ equal to the quantity $\vartheta^\circ(y[t_0], z[t_0])$. Condition (4.1) is then fulfilled for the motion $y[t]$, $z[t]$, $\vartheta[t]$ $(t \geqslant t_0)$ of system (1.1),(1.2),(2.15),(2.16).

Under condition (4.2) we have $\vartheta^\circ(y[t], z[t]) \leqslant \vartheta[t]$. Thus, upon fulfillment of this condition we have the inequality

$$t + \vartheta^\circ(y[t],\ z[t]) \leqslant t + \vartheta[t] \tag{4.3}$$

By conditions (2.15) and (2.16) the quantity $t + \vartheta[t]$ does not increase. For this reason (4.3) yields the inequality

$$t + \vartheta^\circ(y[t],\ z[t]) \leqslant t_0 + \vartheta[t_0] = t_0 + \vartheta^\circ(y[t_0],\ z[t_0]) \tag{4.4}$$

which implies that for some $t = t_* \leqslant t_0 + \vartheta[t_0]$ we have the limit relation

$$\limsup \vartheta^\circ(y[t], z[t]) = 0 \quad \text{as} \quad t \to t_* - 0 \tag{4.5}$$

Since the domain $G^{(1)}[y, \vartheta]$ contracts to a point as $\vartheta \to 0$, and since the domain $G^{(2)}[z, \vartheta]$ lies inside the domain $G^{(1)}[y, \vartheta]$, for $\vartheta = \vartheta^\circ(y, z)$, limit relation (4.5) indicates that encounter of the motions must occur not later than at the instant $t = t_* \leqslant t_0 + \vartheta[t_0]$. It remains for us to verify the fulfillment of condition (4.2).

Let us assume the contrary, i.e. that condition (4.2) is violated before encounter. This means that at some instant $t = t^*$ we have the inequality $\varepsilon^\circ(y[t^*], z[t^*], \vartheta[t^*]) > 0$. Since all the quantities $y[t]$, $z[t]$ and $\vartheta[t]$ in the domain W^ε vary continuously, and since the domain W^ε is open, there exists an instant $t = t_* < t^*$ when the motion $y[t]$, $z[t]$ and $\vartheta[t]$ leaves the domain W_0 for the last time prior to the instant $t = t^*$. Here $\{y[t_*], z[t_*], \vartheta[t_*]\} \in W_0$, since the domain W^ε is open, since the function $\vartheta[t]$ is continuous from the right for all t, and since for $\{y[t], z[t], \vartheta[t]\} \in W^\varepsilon$ this function must (by definition) be continuous from the left as well. But in this case from Property (3) of the function $\vartheta[t]$ we infer that $\varepsilon^\circ(y[t_*], z[t_*], \vartheta[t_*]) = 0$. The quantity $\varepsilon^\circ(y[t], z[t], \vartheta[t])$ varies continuously on the segment $t_* \leqslant t \leqslant t^*$, since the quantities $y[t]$, $z[t]$ and $\vartheta[t]$, vary continuously. Thus, the continuous function $\varepsilon^\circ[t] = \varepsilon^\circ(y[t], z[t], \vartheta[t])$ satisfies the inequality

$$\varepsilon^\circ[t_*] < \varepsilon^\circ[t^*] \qquad (t^* > t_*) \tag{4.6}$$

On the other hand, we can verify that the right-hand upper derivative number of the functions $\varepsilon^\circ[t] = \varepsilon^\circ(y[t], z[t], \vartheta[t])$, which we shall denote by the symbol $(d\varepsilon[t]/dt)_+^{(b)}$, is nonpositive for every $t \in (t_*, t^*)$. Let us show this.

We consider the hyperplane $\varkappa^\circ$ tangent to the boundary of the domain $G^{(1)}[y[t], \vartheta[t], \varepsilon^\circ[t]]$ at the point p°. In the space $\{p\}$ this hyperplane is described by Eq.

$$\varepsilon^\circ[t] + \rho^{(1)}[y[t], \vartheta[t], l^\circ[t]] - l^{\circ\prime}[t]p = 0 \tag{4.7}$$

where $l^\circ[t]$ is the unit vector which satisfies the inequality

$$\varepsilon^\circ[t] + \rho^{(1)}[y[t], \vartheta[t], l^\circ[t]] - \rho^{(2)}[z[t], \vartheta[t], l^\circ[t]] = 0 \tag{4.8}$$

in accordance with (3.6).

By virtue of the uniqueness of the point p° for any other unit vector l, we have the inequality

$$\varepsilon^\circ[t] + \rho^{(1)}[y[t], \vartheta[t], l] - \rho^{(2)}[v[t], \vartheta[t], l] > 0 \tag{4.9}$$

Hence, for any number $\alpha > 0$ there exists a number $\beta(\alpha) > 0$ such that the inequality

$$\varepsilon^\circ[t] + \rho^{(1)}[y[t], \vartheta[t], l] - \rho^{(2)}[z[t], \vartheta[t], l] \geqslant \beta(\alpha) \tag{4.10}$$

holds provided that $\| l \| = 1$ and

$$\| l - l^\circ[t] \| \geqslant \alpha \tag{4.11}$$

Since the quantities $y[t]$, $z[t]$ and $\vartheta[t]$ vary continuously with t, the inequality

$$\varepsilon^\circ[t] + \rho^{(1)}[y[t + \Delta t], \vartheta[t + \Delta t], l] - \rho^{(2)}[z[(t + \Delta t], \vartheta[t + \Delta t], l] \geqslant 0 \tag{4.12}$$

is fulfilled on the motion $\{y[t], z[t], \vartheta[t]\}$ at the instant $t + \Delta t$ whatever the unit vector l (4.11) provided that the quantity Δt is sufficiently small. But in this case we conclude from condition (3.6) that to prove the relation

$$\left(\frac{d\varepsilon^\circ[t]}{dt}\right)_+^{(b)} \leqslant 0 \tag{4.13}$$

it is enough to make use of condition (2.13), where the vector l is restricted by the inequality

$$\| l - l^\circ[t] \| \leqslant \alpha \tag{4.14}$$

Let us assume for the time being that the control $u_*(\tau)$ operates on the segment $t \leqslant \tau \leqslant t + \Delta t$, and that this control aims the motion $y[t]$ at some point q_* lying on the boundary of the domain $G^{(1)}[y[t], \vartheta[t]]$ and is closest to some point p_* lying on the boundary of the domain $G^{(1)}[y[t], \vartheta[t]; \varepsilon^\circ[t]]$ in a small η-neighborhood of the point p° ($\lim \eta(\alpha) = 0$ as $(\alpha \to 0)$. Specifically, if $p_* = p^\circ$, then the control $u_*(\tau)$ coincides with the control which defines the function $u_*[y[t], z[t], \vartheta[t])|$ (see the beginning of Section 2 above). We denote the corresponding motion y by the symbol $y_*(\tau)$.

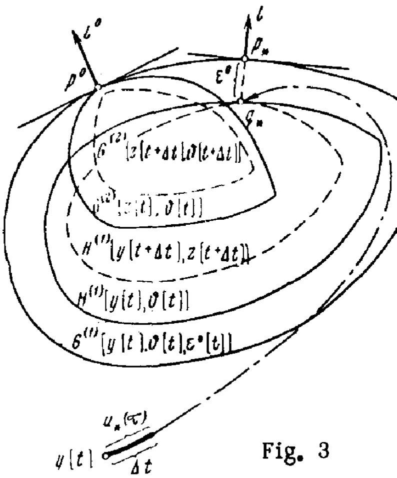

Fig. 3

Under the control $u_*(\tau)$ the point p_* from the domain $G^{(1)}[y[t], \vartheta[t]; \varepsilon^\circ[t]]$ would remain in the domain $G^{(1)}[y_*(t + \Delta t), \vartheta(t + \Delta t); \varepsilon^\circ[t]]$, since the point q_* would clearly remain in the domain $G^{(1)}[y_*(t + \Delta t), \vartheta[t + \Delta t]]$ (Fig. 3). With allowance for condition (2.8), we conclude that this implies fulfillment of the relation

$$\varepsilon^\circ[t] + \rho^{(1)}[y_*(t + \Delta t), \vartheta[t + \Delta t], l] - \rho^{(2)}[z[t + \Delta t], \vartheta[t + \Delta t], l] \geqslant 0$$

which implies precisely the geometric fact that the distance from the domain $G^{(1)}[y_*(t + \Delta t), \vartheta[t + \Delta t]; \varepsilon^\circ[t]]$ to the hyperplane $\varkappa_*$ tangent to this domain $G^{(1)}[y_*(t + \Delta t), \vartheta[t + \Delta t]; \varepsilon^\circ[t]]$ at the point p_* is not larger than the distance from the same hyperplane to the domain $G^{(2)}[z[t + \Delta t], \vartheta[t + \Delta t]]$ (Fig. 3). In fact, however, the control $u[\tau] = u^\circ(y[\tau], z[\tau], \vartheta[\tau])$ operates on the segment $t \leqslant \tau \leqslant t + \Delta t$.

Under our assumption we can verify that with the permissible variation $\delta u(\tau) = u[\tau] - u_*(\tau)$ the controls $u_*(\tau)$ on the segment $t \leqslant \tau \leqslant t + \Delta t$, the domain $G^{(1)}[y[t + \Delta t], \vartheta[t + \Delta t], \varepsilon^\circ[t]]$ will still contain the points p whose deviations from the points p_* in the direction of any vector l from the set (4.14) do not exceed the quantity $\xi[\alpha, \Delta t]\,\Delta t$, where $\xi[\alpha, \Delta t] \to 0$ as $\{\alpha, \Delta t\} \to 0$. This implies the estimate

$$\varepsilon^\circ[t] + \rho^{(1)}[y[t + \Delta t], \vartheta[t + \Delta t], l] - \rho^{(2)}[z[t + \Delta t], \vartheta[t + \Delta t], l] \geqslant$$

$$\geqslant -\Delta t\, \xi[\alpha, \Delta t]$$

From this estimate we infer (by virtue of (2.13)) that relation (4.13) is fulfilled.

Conditions (4.6) and (4.13) are contradictory. The resulting contradiction indicates that in the regular case under consideration the control $u^\circ(y, z, \vartheta)$ constructed in Section 2 ensures encounter of the motions $y[t]$ and $z[t]$ not later than at the instant of absorption $t^\circ = t_0 + \vartheta^\circ(y[t_0], z[t_0])$ provided this instant exists for the initial state of system (1.1),(1.2).

Note. The most interesting motions among $\{y[t], z[t], \vartheta[t]\}$ are those for which the condition

$$\vartheta[t] = \vartheta^\circ(y[t], z[t]) \qquad (t \geqslant t_0) \tag{4.15}$$

is always fulfilled, since this condition ensures the most favorable development of the process from the pursuer's standpoint (within our formulation of the problem). The extremal control $u^\circ(y[t], z[t], \vartheta[t])$ constructed in Section 2 likewise ensures encounter for the class of solutions $\{y[t], z[t], \vartheta[t]\}$ isolated by condition (4.15), but here encounter occurs later than at the instant $t^\circ = \tau + \vartheta^\circ(y[\tau], z[\tau])$ already for $t > \tau$ whatever the instant τ realized prior to encounter. We also note that in the linear case the uniqueness of the point p° is immaterial. All that matters is the uniqueness of the vector $l^\circ[t]$. We also note that the convexity of the domains $G^{(1)}$ and $G^{(2)}$ does not generally play a decisive role either. However, if these domains are nonconvex the condition of regularity of absorption no longer appears natural and is difficult to verify. Finally, we note that the above scheme of constructing the control u° can be transformed in a staightforward way to the problem where the condition of encounter is the inequality

$$\| \{y[t_*] - z[t_*]\}_m \| \leqslant \gamma \qquad (\gamma > 0 - \text{const})$$

The role of the quantity $\vartheta^\circ(y, z)$ in the corresponding construction for u° is played by the time $\vartheta^{(\gamma)}(y, z)$ until the instant of γ-absorption.

BIBLIOGRAPHY

1. Isaacs, R., Differential Games. "Mir" Press, Moscow, 1967.
2. Pontriagin, L. S., On the theory of differential games. Uspekhi Mat. Nauk, Vol. 21, №4, 1966.
3. Pshenichnyi, B. N., On the pursuit problem. Kibernetika, №6, 1967.
4. Krasovskii, N. N., On a problem of tracking. PMM Vol. 27, №2, 1963.
5. Krasovskii, N. N., Motion Control Theory. Linear Systems. "Nauka" Press, Moscow, 1968.
6. Krasovskii, N. N., On the problem of games encounter of motions. Dokl. Akad. Nauk SSSR, Vol. 173, №3, 1967.
7. Filippov, A. F., Differential equations with a discontinuous right side. Mat. Sb., Vol. 51(93), №1, 1960.
8. Krasovskii, N. N. and Subbotin, A. I., The problem of convergence of controlled objects. PMM Vol. 32, №4, 1968.
9. Karlin, S., Mathematical Methods of Games Theory, Programing and Economics. "Mir" Press, Moscow, 1964.
10. Pontriagin, L. S., Boltnianskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., The Mathematical Theory of Optimal Processes. Fizmatgiz, Moscow, 1961.
11. Filippov, A. F., Certain problems of optimal control theory. Vestnik MGU, №2, 1959.

Translated by A. Y.